THIRD NATIONAL STUDENT OLYMPIAD
IN COMPUTER MATHEMATICS
„ACADEMICIAN STEFAN DODUNEKOV“
SOFIA UNIVERSITY ST. KLIMENT OHRIDSKI
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Problems for group A

1. Calculate the value of $\left(\sqrt{2+\sqrt{3}}+\left(2-2\cos\frac{11\pi}{6}\right)^{\frac{1}{2}}\right)\left(\sqrt{2+2\cos\frac{\pi}{6}}-\sqrt{2-\sqrt{3}}\right)^{-1}$.

2. If $F(x,y)=\frac{\sin(x-y)+\cos(x+2y)}{x+2y}$, calculate $F(2F(x,y),F(y,x))$ for $x=4{,}5$ and $y=3{,}8$.

3. Check if the number 2 760 727 302 517 is simple.

4. Factor the polynomial $1-x-7x^2-10x^3-7x^4-x^5+x^6$ into irreducible polynomials with real coefficients.

5. Simplify the expression $2(a+b)^{-1}(ab)^{\frac{1}{2}}\left(1+\frac{1}{4}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2\right)^{\frac{1}{2}}$ if a and b are real numbers.

6. The quadratic equation $x^2-4x+1=0$ has roots x_1, x_2. Calculate the value of $\left(\sqrt{x_1}+\sqrt{x_2}\right)^{-14}$.

7. Prove the identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2k+1}=\frac{2^{2n}(n!)^2}{(2n+1)!}$ for $n\in\mathbb{N}$.

8. Find the integers n, for which n^2+2 is a divisor of $2014n+2$.

9. Find the greatest natural number $n\le 1000$ that can be represented in the form $n=\frac{a^2+b^2}{ab-1}$, where a and b are natural numbers.

10. Calculate f_{2014}, given the sequence $\{f_n\}_{n=1}^{\infty}$ where $f_1=2$, $f_2=1$, $f_{3n}=3f_n$, $f_{3n+1}=3f_n+2$, $f_{3n+2}=3f_n+1$.

11. Calculate $f(A)$ given the function $f(x)=x^{2014}-x^{1989}$ and the matrix $A=\begin{pmatrix}-1 & -2 & 2\\ 4 & 5 & -4\\ 3 & 3 & -2\end{pmatrix}$.

12. Find an irreducible fraction that is a solution of the equation:

$$x=\sqrt{2015-x}\,\sqrt{2014-x}+\sqrt{2014-x}\,\sqrt{2013-x}+\sqrt{2013-x}\,\sqrt{2015-x}.$$

13. Solve the equation $\operatorname{arctg} x-\frac{1}{2}\operatorname{arctg}\frac{2x}{1-x^2}=\frac{\pi}{2}$.

14. Find a solution of the system $\left|\begin{array}{l}24x^3-10x^2y-3xy^2+y^3=0\\ x^2+5x=y^2\end{array}\right.$ such that x has the greatest value.

15. Find the nearest to 10 root of the equation $e^{-x^2} - \cos x = 0$.

16. Find the values of the real parameter m, for which the equation $(1-m)x^3 - 3mx^2 - 3mx + 4 - m = 0$ has three real roots.

17. Find the number of the real solutions of the $\begin{vmatrix} 5x_{k+1} = x_k^5 + 3, \ \ k = 1,2,\ldots,2013 \\ 5x_1 = x_{2014}^5 + 3 \end{vmatrix}$.

18. Calculate $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)$.

19. Calculate the integral $\int_0^{2014} \frac{\ln x\, dx}{\sqrt{2014x - x^2}}$.

20. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and the area of the figure, inside that curve.

21. Let the triangle *ABC* be a right isosceles triangle with hypotenuse *AB* equal to 4. The point *D* lies on a circle with centre *C* and radius 1. Find the smallest possible perimeter of the triangle *ABD*.

22. The lines *CA* and *CB* pass through the point $C(3,-1)$ and are tangent to the ellipse Γ: $2x^2 + 3y^2 + x - y - 5 = 0$ ($A \in \Gamma$, $B \in \Gamma$). Find the area of the triangle ABC.

23. Draw that part of the sphere $x^2 + y^2 + z^2 = 1$ which is inside the cylinder $x^2 + y^2 = x$.

24. Find the volume of the body, defined by the inequality $\left(x^2 + y^2 + z^2 + 8\right)^2 \le 36\left(x^2 + y^2\right)$.

25. Find a natural number N for which the equations $x^2 - 2014x + N = 0$ and $x^2 - 2014x - N = 0$ have integer roots.

26. The sequence $\{p_n\}_{n=1}^{\infty}$ of natural numbers is such that the sequence $\left\{\frac{p_n}{n}\right\}_{n=1}^{\infty}$ is strictly decreasing. Find the least possible value of p_{681} if $p_{2014} = 10000$.

27. Find the number of the squares with area smaller that 201400, which vertices have integer coordinates satisfying the equality $x^4 + x^3y^3 = y^4 + xy$.

28. Find the 2014-th digit of the number 2014^{2014}.

29. The natural number n and the positive numbers $x_1, x_2, \ldots, x_n$ are such that $\sum_{k=1}^{n} x_k = 2014$. Find the greatest value of $\prod_{k=1}^{n} x_k$.

30. Let $\frac{1}{1 - x - 14x^2 + x^3} = \sum_{n=0}^{\infty} a_n x^n$. Calculate the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Each problem is worth 2 points.
All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.