THIRD NATIONAL STUDENT OLYMPIAD
IN COMPUTER MATHEMATICS
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SOFIA UNIVERSITY ST. KLIMENT OHRIDSKI
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Problems for group A

1. Calculate the value of \( \left( \frac{2 + \sqrt{3}}{2} + \left( 2 - 2\cos \frac{11\pi}{6} \right)^{\frac{1}{2}} \right) \left( \frac{2 + 2\cos \frac{\pi}{6} - \sqrt{2 - \sqrt{3}}}{2} \right)^{-1} \).

2. If \( F(x, y) = \frac{\sin(x-y) + \cos(x+2y)}{x+y+2} \), calculate \( F(2F(x,y), F(y,x)) \) for \( x = 4,5 \) and \( y = 3,8 \).

3. Check if the number \( 2760727302517 \) is simple.

4. Factor the polynomial \( 1 - x - 7x^2 - 10x^3 - 7x^4 - x^5 + x^6 \) into irreducible polynomials with real coefficients.

5. Simplify the expression \( 2(a+b)^{-1}(ab)^{\frac{1}{2}} \left( 1 + \frac{1}{4} \left( \sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} \right)^2 \right)^{\frac{1}{2}} \) if \( a \) and \( b \) are real numbers.

6. The quadratic equation \( x^2 - 4x + 1 = 0 \) has roots \( x_1, x_2 \). Calculate the value of \( \left( \sqrt{x_1} + \sqrt{x_2} \right)^{-14} \).

7. Prove the identity \( \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \frac{1}{2k+1} = \frac{2^n (n!)^2}{(2n+1)!} \) for \( n \in \mathbb{N} \).

8. Find the integers \( n \), for which \( n^2 + 2 \) is a divisor of \( 2014n + 2 \).

9. Find the greatest natural number \( n \leq 1000 \) that can be represented in the form \( n = \frac{a^2 + b^2}{ab - 1} \), where \( a \) and \( b \) are natural numbers.

10. Calculate \( f_{2014} \), given the sequence \( \{f_n\}_{n=1}^{\infty} \) where \( f_1 = 2, f_2 = 1, f_{3n} = 3f_n, f_{3n+1} = 3f_n + 2, f_{3n+2} = 3f_n + 1 \).

11. Calculate \( f(A) \) given the function \( f(x) = x^{2014} - x^{1989} \) and the matrix \( A = \begin{pmatrix} -1 & -2 & 2 \\ 4 & 5 & -4 \\ 3 & 3 & -2 \end{pmatrix} \).

12. Find an irreducible fraction that is a solution of the equation:
\[ x = \sqrt{2015 - x} \sqrt{2014 - x} + \sqrt{2014 - x} \sqrt{2013 - x} + \sqrt{2013 - x} \sqrt{2015 - x} \].

13. Solve the equation \( \arctg \frac{x}{2} - \frac{1}{2} \arctg \frac{2x}{1-x^2} = \frac{\pi}{2} \).

14. Find a solution of the system
\[
\begin{align*}
24x^3 - 10x^2y - 3xy^2 + y^3 &= 0 \\
x^2 + 5xy &= y^2
\end{align*}
\]such that \( x \) has the greatest value.
15. Find the nearest to 10 root of the equation \( e^{-x^2} - \cos x = 0 \).

16. Find the values of the real parameter \( m \), for which the equation \((1-m)x^3 - 3mx^2 - 3mx + 4 - m = 0\) has three real roots.

17. Find the number of the real solutions of the equation \( 5x_{k+1} = x_k^5 + 3, \ k = 1, 2, \ldots, 2013 \). \( 5x_1 = x_{2014}^5 + 3 \).

18. Calculate \( \lim_{n \to \infty} \left( n! \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) \).

19. Calculate the integral \( \int_0^{2014} \frac{\ln x \, dx}{\sqrt{2014 x - x^2}} \).

20. Find the length of the curve \( x^2 + y^2 = 1 \) and the area of the figure, inside that curve.

21. Let the triangle \( ABC \) be a right isosceles triangle with hypotenuse \( AB \) equal to 4. The point \( D \) lies on a circle with centre \( C \) and radius 1. Find the smallest possible perimeter of the triangle \( ABD \).

22. The lines \( CA \) and \( CB \) pass through the point \( C(3,-1) \) and are tangent to the ellipse \( \Gamma: 2x^2 + 3y^2 + x - y - 5 = 0 \) \((A \in \Gamma, \ B \in \Gamma)\). Find the area of the triangle \( ABC \).

23. Draw that part of the sphere \( x^2 + y^2 + z^2 = 1 \) which is inside the cylinder \( x^2 + y^2 = x \).

24. Find the volume of the body, defined by the inequality \( (x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2) \).

25. Find a natural number \( N \) for which the equations \( x^2 - 2014x + N = 0 \) and \( x^2 - 2014x - N = 0 \) have integer roots.

26. The sequence \( \{p_n\}_{n=1}^{\infty} \) of natural numbers is such that the sequence \( \left\{ \frac{p_n}{n} \right\}_{n=1}^{\infty} \) is strictly decreasing. Find the least possible value of \( p_{681} \) if \( p_{2014} = 10000 \).

27. Find the number of the squares with area smaller that 201400, which vertices have integer coordinates satisfying the equality \( x^4 + x^3 y^3 = x^4 + xy \).

28. Find the 2014-th digit of the number 2014^{2014}.

29. The natural number \( n \) and the positive numbers \( x_1, x_2, \ldots, x_n \) are such that \( \sum_{k=1}^{n} x_k = 2014 \). Find the greatest value of \( \prod_{k=1}^{n} x_k \).

30. Let \( \frac{1}{1-x-14x^2 + x^3} = \sum_{n=0}^{\infty} a_n x^n \). Calculate the limit \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).

Each problem is worth 2 points.
All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.