Third National Student Olympiad
In Computer Mathematics
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Problems for group B

1. Find a polynomial of forth degree with leading coefficient 1, which zeros are less than the zeros of
\[ x^4 + 4x^3 - 12x^2 - 29x + 56. \]
with 3.

2. Find a polynomial of least possible degree such that for \( x = 1, 2, 3, 4 \) is equal respectively to
\( y = 3, 1, 7, 27 \).

3. Find real \( a \) and \( b \), is the number \( 2 - i \) is a root of the equation
\[ 2x^4 + x^3 - x^2 + ax + b = 0. \]

4. What are the values of \( a \) for which the numbers \( x \), \( y \) and \( z \), derived from the system
\[
\begin{align*}
3x - y + z &= 5 \\
ax + y - z &= 0 \\
x + 2y + az &= 17
\end{align*}
\]
form an arithmetic progression in that order?

5. What are the values of the parameter \( a \) such that the system
\[
\begin{align*}
ax + 3y + z &= 0 \\
(2a - 1)x + 2y + 4z &= 0
\end{align*}
\]
has a non-zero solution?

6. Prove that if \( a, b, c \) and \( d \) are different integers, then \((a - c)^2 + (b - d)^2\) is a divisor of the
determinant
\[
\begin{vmatrix}
a & b & c & d \\
b & c & d & a \\
c & d & a & b \\
d & a & b & c
\end{vmatrix}
\]

7. The point \( C \) lies on the line \( a : x + y = 2014 \). If \( A(2,0) \), \( B(1,4) \) and the area of the triangle \( ABC \) is
2015, then find the coordinates of the point \( C \).

8. Find the area of the figure, defined by the curve \( y = 2 - 4x^2 + 4x^3 - x^4 \), abscise axis and the lines \( x = x_1 \)
and \( x = x_2 \), if \( y \) has local maximum in the points \( x_1 \) and \( x_2 \).

9. Find all four-digit numbers, that are equal to the sum of the forth degrees of their digits.

10. Calculate the limit \( \lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{2 + \cos \frac{\pi}{n}} + \frac{1}{2 + \cos \frac{2\pi}{n}} + \cdots + \frac{1}{2 + \cos \frac{n\pi}{n}} \right) \).

11. Prove the identity
\[
\int_0^{\sin^2 x} \arcsin \sqrt{t} \, dt + \int_0^{\cos^2 x} \arccos \sqrt{t} \, dt = \frac{\pi}{4}.
\]

12. Compare the value of the expression \( \arctg \frac{1}{2} + \arctg \frac{1}{5} + \arctg \frac{1}{8} \) with the number \( \frac{\pi}{4} \).

13. Solve the equation \( \sin x = \log_{10} x \).
14. Calculate the sum \( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \) and compare the value of the integral \( \int_0^{2014} e^{-x} \left( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{2014}}{2014!} \right) \, dx \) with 2014.

15. Find the values of the real parameter \( m \), for which the equation \((1-m)x^3 - 3mx^2 - 3mx + 4 - m = 0\) has three real roots.

16. On the hyperbola \( xy = -1 \) the points \( A_n \) and \( B_n \) are chosen such that their abscises are respectively \( \frac{n}{n+1} \) and \( \frac{n+1}{n} \). Let \( M_n \) be the centre of the circle passing through the points \( A_n, B_n \) and the point \( C(1,-1) \). Find the limit of the sequence of points \( \{M_n\} \) for \( n \to \infty \).

17. Find the intersection point of the planes \( \alpha: 7x - 5y + 2z = 41 = 0 \), \( \beta: 4x + 3y - 11z + 49 = 0 \) and \( \gamma: 2x + 3y + 4z - 20 = 0 \).

18. Find the greatest value of the function \( y = 2\tan x - \tan^2 x \) in the interval \((0, \frac{\pi}{2})\).

19. The local extreme value of the function \( y = \frac{ax + b}{(x - 20)(x - 14)} \) is equal to 1 and is reached for \( x = 2014 \). Find \( a \) and \( b \) and determine the kind of the extreme.

20. Solve the equation \( \arctan x - \frac{1}{2} \arctan \frac{2x}{1-x^2} = \frac{\pi}{2} \).

21. Calculate the limit \( \lim_{n \to \infty} \left( 1 + \frac{1}{n^2} \right) \left( 1 + \frac{2}{n^2} \right) \cdots \left( 1 + \frac{n-1}{n^2} \right) \left( 1 + \frac{n}{n^2} \right) \).

22. Find a differential equation which family of solution curves is \( y = \frac{x^2 - C^2}{2C} \).

23. Tangent lines to the parabola \( y = -x^2 + 5x - 6 \) pass through the point \( M(2,4) \). Find the area of the “curved” triangle, formed by the two tangent lines and the parabola.

24. Two circles are given. The first has a radius 3 and a centre \( M(0,5) \) and the second – a radius 2 and centre \( N(10,7) \). Find a point \( A \) on the first circle, a point \( B \) on the second circle and a point \( C \) on the abscise axis, such that the sum \( AB + BC + AC \) is minimal. Plot the circles and the solution on one drawing.

25. Find the length of the curve \( x^3 + y^3 = 1 \) and the area of the figure defined by that curve.

26. Which is the 2014-th digit of the number \( 2014^{2014} \)?

27. The vertices of a square are used as centres of four circles (but only one fourth of each one is drawn). Find the area of the squared part of the figure.

28. A light beam starting from point to point \( A(-3,0) \) is directed towards point \( B(0,2) \) and is reflected by the ellipse \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \) first at point \( C \), and then at point \( D \). Find the coordinates of the point \( D \).

29. Find the volume of the body, defined by the planes \( y = 1 \) and \( z = 0 \), the parabolic cylinder \( y = x^2 \) and the paraboloid \( z = x^2 + y^2 \).

30. Prove that the inflexion points on the graph of the function \( y = \frac{x + 1}{x^2 + 1} \) lie on a straight line.

Each problem is worth 2 points.

All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.