FOURTH NATIONAL STUDENT OLYMPIAD
IN COMPUTER MATHEMATICS
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Problems for group A

1. Calculate $1+\sqrt{2+\sqrt[3]{3+\sqrt[4]{4+\sqrt[5]{5}}}}$.
2. Calculate the value of the expression $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot \cdots \cdot\frac{98}{99}\cdot\frac{100}{101}$.
3. Simplify the expression $\left(\frac{a-\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}-\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right):\frac{4\sqrt{a^4-a^2b^2}}{(5b)^2}$, where $a>b>0$ are real numbers.
4. Calculate the value of the expression $\frac{1}{1+x+x.y}+\frac{1}{1+y+y.z}+\frac{1}{1+z+z.x}$ for all values in the domain of the variables, if $x.y.z=1$.
5. Find the real zeros of the polynomial $-1+x-x^2+3x^3-5x^4+x^5$.
6. Determine the numbers a and b so that the polynomial ax^4+bx^2+1 is divisible to $(x-2)^2$.
7. Solve the equation $z^2-(2+i)z-1+7i=0$.
8. Find the real roots of the equation $\sqrt{x^3-100}=\sqrt[3]{x^2+100}$.
9. Solve the equation $e^{2x}+\sin 3x=4$.
10. Find the roots of the equation $10e^{-0,1x^2}=\sqrt{2\pi+x}+\sin 2x.$
11. Find the greatest integer $n,$ for which $2015!$ is divisible without remainder to 2015^n.
12. What are the integers $n,$ for which the number $n^4+8n^3+17n^2+8n+1$ is simple?
13. Which is the last digit in the decimal representation of the number $\frac{2015}{5^{2015}}$?
14. Find all natural numbers $M,$ for which $M^{2015}-1$ is divisible to 10000.
15. Among the first 50000 simple numbers find those that contain in their decimal representation the sequence 2015 (the smallest number with this property is 120157).
16. How many times each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 can be found in the first 2015 digits of the Napier's constant e?
17. Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix}-13 & -32 & -61\\ 10 & 23 & 38\\ -2 & -4 & -4\end{pmatrix}$.
18. The square matrices $A_n=(a_{ij}),\ 1\le i\le n,\ 1\le j\le n$ are defined by the equalities $a_{ij}=i-j^2$. Determine the values of $n\in\{2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$ for which the determinant of the matrix матрицата A_n is equal to 0.

19. The points $A(4,0)$ and $B(0,3)$, and the ellipse $E\ :\ 12\,x^2 + 16\,y^2 = 7.$ are given in a Cartesian coordinate system on a plane. Find the coordinates of the point $C \in E,$ for which the arear of ΔABC is smallest and calculate that area.

20. Which is the least natural number $n,$ for which the sum $\sum_{k=1}^{n} \frac{30^k}{k!} - \frac{20^k}{(k-1)!}$ is greater than 10^{12}? And for which n the sum is greater than 10^{13}? Can this sum become greater than 10^{14} for any n?

21. Find the greatest value of the function $f(x) = \begin{vmatrix} 2 & x & x & x \\ x & 0 & x & x \\ x & x & 1 & x \\ x & x & x & 5 \end{vmatrix}$ over the interval $[0, 2015].$

22. What are the values of the real parameter m such that the equation $(2-m)x^3 - 3mx^2 - 3mx + 2 - m = 0$ has a double root?

23. Prove that for every two positive numbers a and b the inequality $\sqrt[3]{4}\left(a^3 + b^3\right) \geq 3a^2b.$ holds.

24. Find the greatest among the numbers $\sqrt{\frac{2}{2015}},\ \sqrt[3]{\frac{3}{2015}},\ \sqrt[4]{\frac{4}{2015}},\ \sqrt[5]{\frac{5}{2015}},\ \ldots,\ \sqrt[201520152015]{\frac{201520152015}{2015}}.$

25. Calculate the limit $\lim_{n\to\infty} \frac{1}{n}\left(\frac{\left(1+\cos\frac{\pi}{n}\right)^2}{1+\sin\frac{\pi}{n}} + \frac{\left(1+\cos\frac{2\pi}{n}\right)^2}{1+\sin\frac{2\pi}{n}} + \cdots + \frac{\left(1+\cos\frac{n\pi}{n}\right)^2}{1+\sin\frac{n\pi}{n}} \right).$

26. Solve the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{12}$ за $x > \sqrt{2}$.

27. Calculate the length of the curve $\ell : \begin{cases} x = \frac{2}{3}t^3 + \frac{t^2}{2} \\ y = -\frac{4}{3}t^3 + \frac{t^2}{2} \\ z = \frac{t^3}{3} + t^2 \end{cases}$ for $t \in [0; 1].$

28. Calculate the area of the figure defined with the inequalities $x^4 + y^4 \leq 1$ and $x^2 \leq y.$

29. Find the area of the figure, bounded by the curve $y = \ln x$, its tangent in the point $(e,1)$ and Ox axis.

30. Find the volume of the body formed after rotation of the figure, bounded by the curves $y = e^{-2x} - 1,\ y = e^{-x} + 1$ и $x = 0.$ around Ox axis.

Each problem is worth 2 points.
All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.