FOURTH NATIONAL STUDENT OLYMPIAD
IN COMPUTER MATHEMATICS
„ACADEMICIAN STEFAN DODUNEKOVT”
BURGAS FREE UNIVERSITY
13 – 15 NOVEMBER 2015

Problems for group B

1. Calculate the value of the expression $\sqrt{5 + \sqrt{3}} + \sqrt{5 + \sqrt{3}}$.

2. Calculate the value of the expression $1 + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{3} + \frac{1}{\sqrt{4} + \frac{1}{\sqrt{5} + \frac{1}{\sqrt{6}}}}}}$.

3. Calculate
$$\sqrt{2015} + \frac{1}{\sqrt{2015} + \sqrt{2015} + \sqrt{2015} + \sqrt{2015} + \sqrt{2015}} - \sqrt{2015} + \sqrt{2015} + \sqrt{2015} + \sqrt{2015} + \sqrt{2015}$$.

4. Calculate the sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{50^2}$.

5. Compare the Napier’s constant $e$ with the value of the expression $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \cdots + \frac{2014}{2015}$. Is it true that $\frac{1}{\sqrt{n^2} < \sqrt{e}}$?

6. Show that for every natural number $n$ the inequality $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \sqrt{e}$ is true.

7. Simplify the expression $\frac{x^4 + 7x^3 + 21x^2 + 63x + 108}{x^4 + 4x^3 - 81x^2 - 270x}$.

8. Show that if 1 is added to the product of 4 consecutive integers, a perfect square is obtained.

9. Find the roots of the equation $x^2 - 5x + 2 = 0$.

10. Solve the equation $\sqrt{1 + x\sqrt{x^2 + 24}} = x + 1$.

11. How many real roots has the equation $100 \cos(10x^2) = x^4 + 1$ in the interval $[0, \frac{\pi}{2}]$?

12. Find the closest to 8 root of the equation $\cos x - \tan x = 0$.

13. Find the number of divisors of the number $1691146334800$.

14. Determine for how many integers $n$ from 1 to 2015 the number $3n^5 + 11$ is simple.

15. How many times each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 can be found in the first 2015 digits of the number $\pi$?

16. Construct a polynomial with lowest degree with real coefficients that has simple roots 2, 3 and $1 + i$, and a double root 1.

17. Given the function $f(x) = x^{2015} - 2^7 \cdot x^{1944}$ and the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 5 & -3 & -15 \\ -1 & 1 & 5 \end{pmatrix}$, calculate $f(A)$.
18. Prove that if $0 < a \leq b \leq c$, then
\[
\begin{vmatrix}
    a & b & c \\
    a^3 & b^3 & c^3 \\
    \frac{1}{b^3 + c^3} & \frac{1}{c^3 + a^3} & \frac{1}{a^3 + b^3}
\end{vmatrix} \geq 0.
\]

19. Solve the system
\[
\begin{align*}
    2x + (2-a)y &= 6 \\
    (a+1)x + y &= 3
\end{align*}
\]
where $a$ is a real parameter.

20. Find the equation of the sphere that passes through the points $A(3,1,5)$, $B(4,-8,1)$ and $C(-5,1,-3)$, if it is known that its center lies on the plane $2x + y - z - 1 = 0$.

21. For the function $f(n)$ is known that $f(1) = 2$ and $f(n) = 3f^2(n-1) - 4$ for $n > 1$. Calculate $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$.

22. Calculate the limit $\lim_{x \to a} \frac{x^n - a^n - n.a^{n-1}(x-a)}{(x-a)^2}$, $n \in \mathbb{N}$, $a \in \mathbb{R}$.

23. Determine the intervals of monotony and the local extrema of the function $f(x) = x^2 - 5x - \frac{2}{3}$.

24. For which natural number $n$ the expression $\frac{n^2}{1,001^n}$ becomes greatest?

25. Prove that the equality $\arctg \sqrt{\frac{1+x}{1-x}} = \frac{\pi}{4} + \frac{1}{2} \arcsin x$ holds for every $x \in (-1,1)$.

26. Calculate $\int_0^{2015} \frac{x}{\sin x + 2015} dx$.

27. Calculate the length of the curve, defined by the graph of the function $y = \ln \left( \frac{e^x + 1}{e^x - 1} \right)$ in the interval $1 \leq x \leq 2$.

28. Find the area of the shaded figure.

29. Find the area of the figure, bounded by the curves $y = \ln (x + 2)$, $y = 2 \ln x$ and $y = 0$.

30. Let $S$ be the area of the figure, bounded by the abscissa axis and the graph of the function $f(x) = (x + 2)(0.5 - x)$. Find the values of the real parameter $b$ for which the rectangular with sides $x = -2$, $y = 0$, $x = 0.5$ and $y = f(b)$ has an area $S$.

Each problem is worth 2 points.
All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.