TENTH NATIONAL STUDENT OLYMPIAD IN COMPUTER MATHEMATICS "ACADEMISIAN STEFAN DODUNEKOV"

"ANGEL KUNCHEV" UNIVERSITY OF RUSE

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PROBLEMS FOR GROUP A

1. Let $f_1(x) = \sin x$ and $f_k(x) = f\left(f_{k-1}(x)\right),\ k = 2, 3, \ldots$ Plot the graphs of the functions $f_1(x)$, $f_{20}(x)$, $f_{40}(x)$ и $f_{50}(x)$ for $0 \le x \le 6\pi$.

2. Find a polynomial p(x) of third degree with rational coefficients, such that

$$p\left(\sqrt{2}+\sqrt{3}\right) = \sqrt{2}.$$

3. Find the number of the solutions in natural numbers (less than 10) of the equation

$$x^{2023} + y^{2023} \equiv z^{2023} (mod\ 10).$$

4. Compute $f(20)$, $f(23)$ and $f(2023)$, if

$$f(n) = \begin{cases} n-3 & \text{for } n \ge 2320, \\ f(f(n+6)) & \text{for } n < 2320. \end{cases}$$

5. It is well-known that every natural number n can be represented in a unique way in a factorial number system: $n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1!$.
For example, $46_{10} = 1.4! + 3.3! + 2.2! + 0.1! = 1320_!$
Find the coefficients of the number 2023 in the factorial number system. Find the last 9 coefficients $a_9, a_8, \ldots, a_1$ of the number 10^{2023} in the factorial number system.

6. Find the largest natural number n, for which 2023! is divisible by 2023^n.

7. Find the three smallest positive solutions of the equation $\operatorname{tg} x = x + 1$.

8. Compute the limit $L = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{k\sqrt{k+n}}{k.n+1}$.

9. Compute the area of the intersection of the interiors of the ellipses

$$\left(\frac{x}{20}\right)^2 + \left(\frac{y}{23}\right)^2 = 1, \qquad \left(\frac{x}{23}\right)^2 + \left(\frac{y}{20}\right)^2 = 1.$$

10. Find all prime numbers p, for which the first composite number in the sequence

$a_n = n^2 + n + p,\ n = 0, 1, 2, ..., 2023$, is the integer a_{p-1}.

11. Given an ellipse with equation $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$. The segment AB has mid-point $M(2,3)$, and its end-points lie on the ellipse. Find the length of the segment AB.

12. How many are the natural numbers $n \leq 2023$, for which the number $n^{2023} + 2023$ is a prime number?

13. Solve the system of equations $x^y = 20,\ y^x = 23$.

14. Let p and q are natural numbers. How many of the equations $x^2 - px - q = 0$ have a positive real root not exseeding 6?

15. Given the matrix $A = \begin{pmatrix} 2 & 0 & 2 & 3 \\ 3 & 2 & 0 & 2 \\ 2 & 3 & 2 & 0 \\ 0 & 2 & 3 & 2 \end{pmatrix}$. Find the eigenvalues of the matrix $B = A^4(2A^T - 3E)$, where the matrix A^T is the transpose of A and E is the identity matrix of order 4.

16. How many consecutive zeros are there after the decimal points in the decimal representation of the number $\left(2 - \sqrt{3}\right)^{100}$?

17. Find the real roots of the equation $\det(Ax + E) = 0$, where A is a square matrix of order 23 with elements $a_{ij} = i.j,\ 1 \leq i \leq 23,\ 1 \leq j \leq 23$, and E is the identity matrix of order 23.

18. Ivan decided to generate a sequence of prime numbers based on Euclid's proof that the number of prime numbers is infinite. Ivan put $p_1 = 2$ and iteratively, upon finding $p_1, p_2, \ldots, p_n$, defined p_{n+1} as the smallest prime divisor of the integer $1 + p_1 p_2 \ldots p_n$. Find p_{15}.

19. Find the number of the integers in the sequence $a_n = n^2 + 20^n + 23^n$ for $n = 1, 2, \ldots, 2023$, that are multiples of 2023.

20. Find the centroid of the set of all points M with integer coordinates such that M $\in \Omega \backslash \Phi$, where:
- Ω is a right circular cone with vertex at $O(0,0,0)$, base radius $r = 8$ and height OH, where $H(0,10,0)$;
- Φ is a sphere with center at the point $C(1,7,-1)$ and radius $\rho = 3$.

21. Find the integers $k,\ 6 \leq k \leq 100$, such that both integers 2023_k and 2025_k (k-base) are prime?

22. Prove that a_k is an exact square for every natural number k, where

$$a_k = \sum_{n=0}^{k} \frac{\sum_{i=1}^{3n+1} i^2}{\sum_{i=1}^{3n+1} i}$$

23. Given a square ABCD with vertices $A(-1,-1),\ B(1,-1),\ C(1,1),\ D(-1,1)$. Find the equation of a line making an angle of 60 degrees with the positive direction of the abscissa, intersecting the side AB and dividing the square into parts whose areas are in the ratio 1:7.

24. Find the asymptotes of the function $f(x) = (3x - 7)\ arctg\left(\frac{x}{2} + 1\right)$.

25. Solve the equation $x + |\sin(x - 1)| = 5$.

26. We call a natural number *n* an exact power if there exist natural numbers *a* and *b* such that $a \geq 2,\ b \geq 2$ and $n = a^b$. Find the number of exact powers that are less than 20232023.

27. Plot the curve defined parametrically:

$$x(t) = \int_0^t \sin\left(\frac{t^2}{2}\right) dt, \quad y(t) = \int_0^t \cos\left(\frac{t^2}{2}\right) dt, \qquad -12 \leq t \leq 12.$$

28. Solve the equation $x^2 + x + 1 = 3y^2$ in positive integers less than 2023.

29. Given a tetrahedron *ABCD* with vertices $A(10,4,1),\ B(1,12,3),\ C(4,2,11)$ and $D(0,3,9)$. Find the coordinates of the point *M* for which the sum of the squares of the lengths of the perpendiculars from *M* to the sides of the tetrahedron is the smallest.

30. Plot the curves with equations in polar coordinates $r(t) = \sin(\frac{k+1}{k}t)$ for $k = 1, 4, 7, 10$.

Time for work: 4 hours — Each problem is evaluated with 2 points.

All numerical calculations should be performed with the implied accuracy of the computer mathematics system used.